$$= \quad c^x \frac{(b^x + a^x)\ln c - (a^x \ln a + b^x \ln b)}{(b^x + a^x)^2}$$

$$= \quad c^x \frac{b^x(\ln c - \ln b) + a^x(\ln c - \ln a)}{(b^x + a^x)^2}.$$

The ln function is increasing, so $\ln c > \ln b$ and $\ln c > \ln a$; thus we see that the derivative is positive. Hence the function f is increasing, so $\frac{1}{2} = \text{f}(0) \le f(x)$ for $x \ge 0$. Because the derivative is strictly positive, the function f actually grows: so $f(x) > \frac{1}{2}$ for $x > 0$.

To verify the inequality of the problem, we note that the tangent function is increasing, so in each summand the tangent term in the numerator is larger that each tangent term in the denominator. Hence we can apply the lemma to each of the three summands, forcing the sum $\ge \frac{3}{2}$. Note that equality holds if and only if $n = 0$.

Comment: We can apply the lemma to obtain some ugly inequalities which are clearly true:

$$\frac{3^n}{1^n + 2^n} + \frac{4^n}{2^n + 3^n} + \frac{5^n}{3^n + 4^n} + \cdots + \frac{(n+2)^n}{n^n + (n+1)^n} \quad \ge \quad \frac{n}{2}, \text{ and}$$

$$\frac{[(n+2)!]^n}{[n!]^n + [(n+1)!]^n} \quad \ge \quad \frac{1}{2}.$$

Also solved by Arkady Alt, San Jose, CA (two solutions); Paul M. Harms, North Newton, KS; Kee-Wai Lau, Hong Kong, China; Paolo Perfetti, Department of Mathematics, University of Tor Vergata, Rome, Italy; Angel Plaza, University of Las Palmas de Gran Canaria, Spain; Ioannis D. Sfikas, National and Kapodistrian University of Athens, Greece; Albert Stadler, Herrliberg, Switzerland, and the proposers

5483: *Proposed by D.M. Bătinetu-Giurgiu, "Matei Basarab" National College, Bucharest and Neculai Stanciu, "George Emil Palade" School Buzău, Romania*

If $a, b > 0$, and $x \in \left(0, \frac{\pi}{2}\right)$ then show that

$$(i) \qquad (a+b)\cdot\frac{\sin x}{x} + \frac{2ab}{a+b}\cdot\frac{\tan x}{x} \ge \frac{6ab}{a+b}.$$

$$(ii) \qquad a\cdot\tan x + \ b\cdot\sin x > 2x\sqrt{ab}.$$

Solution 1 by Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy

Proof of (i).

The AHM yields

$$a+b \ge \frac{4}{\frac{1}{a}+\frac{1}{b}} \iff (a+b)^2 \ge 4ab$$

and then

$$(a+b)\cdot\frac{\sin x}{x}+\frac{2ab}{a+b}\cdot\frac{\tan x}{x} \ge \frac{4ab}{a+b}\cdot\frac{\sin x}{x}+\frac{2ab}{a+b}\cdot\frac{\tan x}{x}$$

Thus we prove

$$\frac{4ab}{a+b}\cdot\sin x+\frac{2ab}{a+b}\cdot\tan x-\frac{6ab}{a+b}x \ge 0$$

This is equivalent to

$$f(x) \doteq 4\sin x+2\tan x-6x \ge 0$$

$$f'(x)=4\cos x+\frac{2}{\cos^2 x}-6$$

$$f''(x)=-4\sin x+\frac{4\sin x}{\cos^3 x}=4\sin x\left(\frac{4}{\cos^3 x}-1\right)>0$$

via $\cos x \in (0,1)$ for $0<x<\pi/2$. Since $f'(0)=f(0)=0$ we get $f(x)\ge 0$.

Proof of (ii).
Let

$$f(x)=a\cdot\tan x+b\cdot\sin x-2x\sqrt{ab}, \quad f(0)=0$$

$$f'(x)=\frac{a}{\cos^2 x}+b\cos x-2\sqrt{ab} \ge \frac{a}{\cos x}+b\cos x-2\sqrt{ab} \ge 2\sqrt{\frac{ab\cos x}{\cos x}}-2\sqrt{ab}=0$$

and this concludes the proof.

Solution 2 by Arkady Alt, San Jose, CA

(i) First we will prove inequality
$\tan x+2\sin x>3x \iff \dfrac{\tan x}{x}+\dfrac{2\sin x}{x}>3,\ x\in(0,\pi/2)$.
Let $h(x):=\tan x+2\sin x-3x,\ x\in(0,\pi/2)$. Since $h'(x)=\dfrac{1}{\cos^2 x}+2\cos x-3=\dfrac{(2\cos x+1)(1-\cos x)^2}{\cos^2 x}>0,\ x\in(0,\pi/2)$ then $h(x)>h(0)=0$.
Hence, $(a+b)\dfrac{\sin x}{x}+\dfrac{2ab}{a+b}\cdot\dfrac{\tan x}{x}>(a+b)\sin x+\dfrac{2ab}{a+b}\cdot\left(3-\dfrac{2\sin x}{x}\right)=\dfrac{\sin x}{x}\left(a+b-\dfrac{4ab}{a+b}\right)+\dfrac{6ab}{a+b}=\dfrac{\sin x}{x}\cdot\dfrac{(a-b)^2}{a+b}+\dfrac{6ab}{a+b}\ge\dfrac{6ab}{a+b}$.
(ii) Let $h(x):=a\tan x+b\sin x-2x\sqrt{ab}$. Since $h'(x)=\dfrac{a}{\cos^2 x}+b\cos x-2\sqrt{ab}\ge 2\sqrt{\dfrac{a}{\cos^2 x}\cdot b\cos x}-2\sqrt{ab}=2\sqrt{ab}\cdot\dfrac{1-\sqrt{\cos x}}{\sqrt{\cos x}}>0$ then $h(x)>h(0)=0 \iff a\tan x+b\sin x>2x\sqrt{ab}$.

Solution 3 by Kee-Wai Lau, Hong Kong, China